Which of these series do you think converges?

(That is, à priori – we will cover precise criteria for each case in the next slides.)

(A) $\sum_{n=1}^{\infty} \frac{1}{n}$ $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (diverges)

(B) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$, $\frac{1}{(n+1)(n+3)} = \frac{1}{2}\left[\frac{1}{(n+1)} - \frac{1}{(n+3)}\right]$

$\rightarrow$ Converge (telescopes)

(C) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ $= 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$ $= \frac{1}{1-2/3} = 3$

(geometric series: $r = \frac{2}{3} < 1$ converges)

(D) None of these

The Harmonic Series

The ***Harmonic Series*** $\sum_{n=1}^{\infty} \frac{1}{n}$ ***diverges*!**

~~*(Recall that we saw a proof of this fact in the Week 5 slides!)*~~

→ later: p-series with $p=1$:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges when $p>1$, diverges otherwise

Telescoping Series

- A telescoping series has the form:

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)}$$

- These series ***converge***.
- To find the sum, use *partial fractions*.

An Example:

Evaluate the following sum: $$S = \sum_{k=0}^{\infty} \frac{\boxed{(-1)^{k+1} 2^k}(3k+1)}{(k+1)(k+3)}$$

→ apply partial fractions:

$$\frac{3k+1}{(k+1)(k+3)} = \frac{A}{k+1} + \frac{B}{k+3}$$

$$3k+1 = A(k+3) + B(k+1)$$

when $k=-1$: $-2 = 2A \longrightarrow A = -1$

when $k=-3$: $-8 = -2B \longrightarrow B = 4$

$$\sum_{k=0}^{\infty} (-1)^{k+1} 2^k \left[\frac{-1}{(k+1)} + \frac{4}{(k+3)} \right] \quad (*)$$

$$S^{(*)} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{(k+1)} - \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k 2^{k+2}}{(k+3)}}_{S_2}$$

$$= \underbrace{1 - \frac{2}{2}}_{=0} + \underbrace{\sum_{k=2}^{\infty} \frac{(-1)^k 2^k}{(k+1)}}_{\text{shift the indices}} - S_2$$

shift the indices $m = k-2$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m 2^{m+2}}{(m+3)} - S_2 = 0$$

Geometric Series

- A geometric series has the form:

$$G = \sum_{n=0}^{\infty} r^n$$

$-1 < r < 1$

- It ***converges* when $|r|<1$** and ***diverges*** otherwise.
- If $|r|<1$, the sum is:

$$\boxed{G = \frac{1}{1-r}.}$$

$$S_N = \sum_{k=0}^{N} r^k = \frac{r^{N+1}-1}{r-1}, \quad |r|<1$$

$$G = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left[\frac{\overset{\to 0}{r^{N+1}}-1}{r-1}\right]$$

$$= \frac{-1}{r-1} = \frac{1}{1-r}$$

Example 1.1:

Sum the series: $$\sum_{n=2}^{\infty}\frac{5^{n-1}+3\cdot 2^{3n}}{9^n} = S$$

$$S = \sum_{N=2}^{\infty}\frac{5^{N-1}}{9^N} + 3\cdot\sum_{N=2}^{\infty}\frac{8^N}{9^N}$$

$m = N-2$ $\qquad$ $m = N-2$

$$= \sum_{m=0}^{\infty}\frac{5^{m+1}}{9^{m+2}} + 3\cdot\sum_{m=0}^{\infty}\frac{8^{m+2}}{9^{m+2}}$$

$$= \frac{5}{81} \cdot \underbrace{\sum_{m=0}^{\infty} \left(\frac{5}{9}\right)^m}_{r = \frac{5}{9} < 1} + \frac{3 \cdot 64}{81} \cdot \underbrace{\sum_{m=0}^{\infty} \left(\frac{8}{9}\right)^m}_{r = \frac{8}{9} < 1}$$

$$= \frac{5}{81} \cdot \underbrace{\frac{1}{1-\frac{5}{9}}}_{\frac{9}{4}} + \frac{3 \cdot 64}{81} \cdot \underbrace{\frac{1}{1-\frac{8}{9}}}_{9}$$

Example 1.2:

Use series to write the decimal

$C =$ 1.42424242… as a *rational number*.

$$1.\overline{42}$$

$$C = 1 + \frac{42}{100} + \frac{42}{100^2} + \frac{42}{100^3} + \cdots$$

$$= 1 + 42 \cdot \sum_{N=1}^{\infty} \frac{1}{100^N}$$

change variable

$m = N - 1$ (shift the index)

$$= 1 + 42 \cdot \sum_{m=0}^{\infty} \frac{1}{100^{m+1}}$$

$$= 1 + \frac{42}{100} \cdot \sum_{m=0}^{\infty} \left(\frac{1}{100}\right)^{m}$$

$r = \frac{1}{100} < 1$

geometric series: $\frac{1}{1-r}$

$$= 1 + \frac{42}{100} \cdot \frac{1}{1 - \frac{1}{100}}$$

$$= 1 + \frac{42}{100} \cdot \frac{1}{\frac{99}{100}} = 1 + \frac{42}{99}$$

Divergence (n^{th} term) Test

Given $\sum_{n=0}^{\infty} a_n$, first find $\lim_{n\to\infty} a_n$.

If $\lim_{n\to\infty} a_n \neq 0$, then the series **DIVERGES**!

Otherwise, the test is *INCONCLUSIVE*

and you must try another test.

<u>Important:</u> *n^{th} term test only tests for divergence!!*

- If the limit of the terms is <u>equal to 0</u>, you do not have enough information!
- For instance: $\sum_{N=1}^{\infty} \frac{1}{N}$
  - The harmonic series, the terms go to 0 but the series diverges!
  - Telescoping series, the terms go to 0 and these series converge!
- So… in order to converge, we need the limit to go to zero, but it is NOT a sufficient condition to determine convergence!

Example A:

Does the series diverge by the n^{th} term test?

$$\sum_{k=1}^{\infty} \underbrace{\left(1+\frac{3}{k}\right)^k}_{a_k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(1+\frac{3}{k}\right)^k = e^3 \neq 0$$

→ so the diverges by the nth term test

Example B:

Does the series diverge by the n[th] term test?

$$\underbrace{\sum_{k=2}^{\infty} \frac{3k}{5k-7}}_{a_k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{3k}{5k-7} = \frac{3}{5} \neq 0$$

$$= \lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{k}} \cdot \frac{3k}{5k-7}$$

$$= \lim_{k\to\infty} \frac{3}{5-7/k} = \frac{3}{5}$$

→ so by the n^{th} term test, the series diverges

Example C:

Does the series diverge by the nth term test?

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{\sqrt{k^3+6}}}_{a_k}$$

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{\sqrt{k^3+6}}$$

$$\sqrt{k^3+6} = \sqrt{k^3\left(1+\frac{6}{k^3}\right)}$$

$$= \lim_{k \to \infty} \frac{1}{k^{3/2}\sqrt{1+\frac{6}{k^3}}} = 0$$

$\longrightarrow$ So the nth term test is inconclusive

Which statement is always true?

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then:}$$

Sequence – a_n

Series – $\sum_{n=0}^{\infty} a_n$

A. ~~The series converges.~~ (counterexample: harmonic series)

✓B. The sequence converges.

C. ~~The sequence of partial sums converges.~~ (same as the series conv.)

D. ~~The series diverges.~~

Sequence – a_n

Series – $\sum_{n=0}^{\infty} a_n$

→ a point of the n^{th} term test:

Sequence converges:

$$L = \lim_{n\to\infty} a_n$$

→ if $L \neq 0$, the series must diverge

→ if $L = 0$, then we need more information to tell whether the series conv. or div.

Some Convergence Theorems

(1) If $\overset{A}{\sum a_n}$ and $\overset{B}{\sum b_n}$ both converge, then

$\sum (a_n \pm b_n)$ also converges.

if A, B are both finite, then $A \pm B$ is finite

(2) If $\sum \overset{A}{a_n}$ converges, then $\sum ca_n$ also

converges for any $c \in \Re$. $-\infty < c < +\infty$

$\mathbb{R}$ = reals

if A is finite, and c is any real number, then cA is finite

(3) If $\sum_{n=j}^{\infty} a_n$ converges, so does $\sum_{n=0}^{\infty} a_n$.

$J = 0, 1, 2, 3, \ldots$

Math 1552

Sections 10.3, 10.4 and 10.5 : Convergence Tests for Infinite Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Learning Goals

"tests"

- Learn how to apply the integral, comparison, limit comparison, ratio and root series to determine whether an infinite series converges or diverges
- Learn when to apply which test
- Summarize the results into a formal mathematical justification

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES. (important to remember)

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- Telescoping series CONVERGE. Find the sum using partial fraction decompositions.

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- Telescoping series CONVERGE. Find the sum using partial fraction decompositions.
- A geometric series

$$\sum_{k=0}^{\infty} r^k \qquad \begin{array}{l} \textit{converges} \text{ to } \dfrac{1}{1-r} \text{ when } |r|<1 \\ \textit{diverges} \text{ when } |\mathrm{r}| \geq 1 \end{array}$$

$-1 < r < 1$

when $r \leq -1$ or $r \geq 1$

Divergence (n^{th} term) Test

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